# Measuring incompatibility and clustering quantum observables with a quantum switch


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## Mutual Eigenspace Disturbance (MED)

We quantify incompatibility of two observables $A$ and $B$ given by the projector sets $\mathbf{P} = (P_i)_{i=1}^{k_A}$ and $\mathbf{Q} = (Q_j)_{j=1}^{k_B}$ acting in a $d$-dimensional Hilbert space by **Mutual Eigenspace Disturbance (MED):**

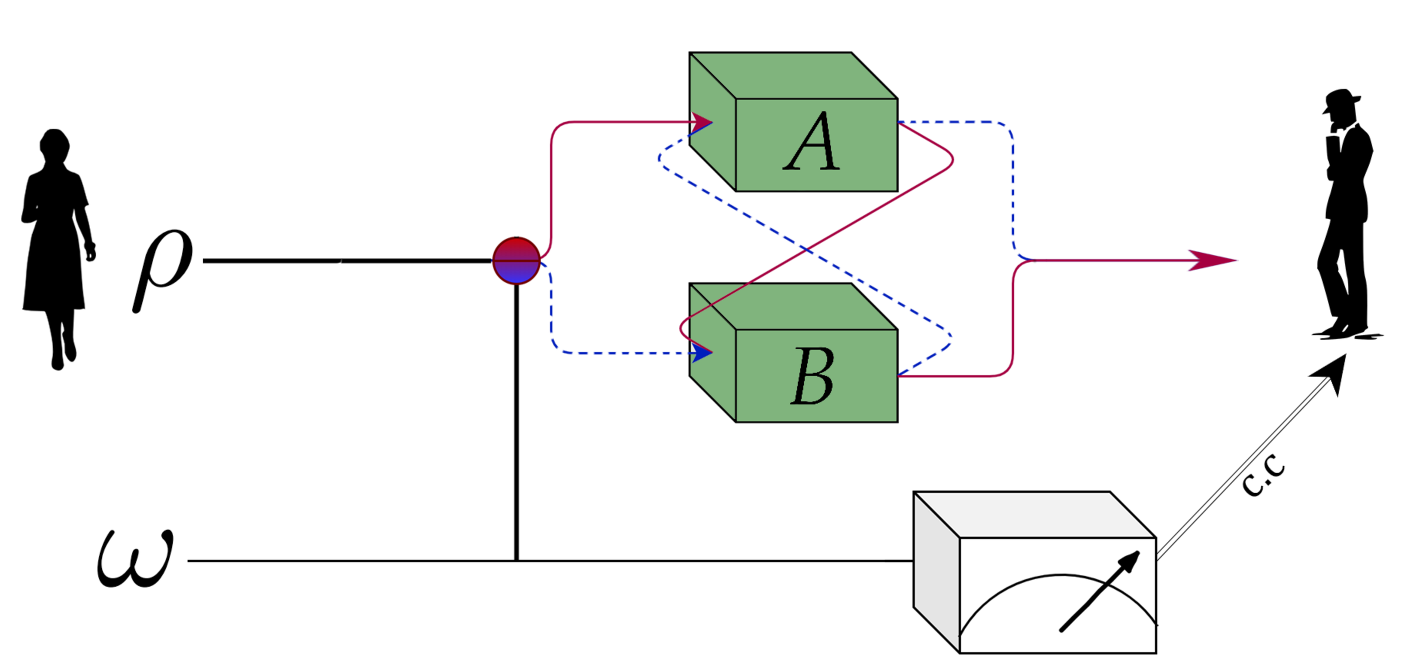
$$\mathrm{MED}(A,B) = \sqrt{1 - \mathsf{Prob}(A,B)}\,, \qquad (1)$$
$$\mathsf{Prob}(A,B) = \frac{1}{d}\sum_{ij} \mathrm{Tr}[P_i Q_j P_i Q_j]\,.$$

**Intuition:** commuting $A$ and $B$ have a common set of eigenstates, and a measurement of e.g. $A$ leaves the eigenspaces of $B$ invariant. The violation of this condition indicates their incompatibility.

## MED is a measure of incompatibility: properties

1) *symmetric and nonnegative*, $\mathrm{MED}(A,B) = \mathrm{MED}(B,A) \geq 0$ for any $A$ and $B$,
2) *faithful*, $\mathrm{MED}(A,B) > 0$ if and only if $A$ and $B$ are incompatible,
3) *maximal for maximally complementary observables*,
4) *decreasing under coarse-graining*,
5) *a metric on von Neumann measurements*,
6) *robust to noise*.

## Experimental setup: Quantum SWITCH




Output of the quantum SWITCH:

$$\mathcal{S}_{A,B}(\rho\otimes\omega) = \frac{1}{4}\sum_{ij}\Big(\{P_i,Q_j\}\rho\{P_i,Q_j\}^\dagger\otimes\omega + \{P_i,Q_j\}\rho[P_i,Q_j]^\dagger\otimes\omega Z + [P_i,Q_j]\rho\{P_i,Q_j\}^\dagger\otimes Z\omega + [P_i,Q_j]\rho[P_i,Q_j]^\dagger\otimes Z\omega Z\Big).$$

It can be seen as a quantum channel with Kraus operators

$$S_{ij} = \underbrace{P_i Q_j\otimes|0\rangle\langle 0|}_{\text{order } B\to A} + \underbrace{Q_j P_i\otimes|1\rangle\langle 1|}_{\text{order } A\to B}$$

## Effective estimation of MED of unknown observables

We choose $\omega = |+\rangle\langle +|$ and perform a measurement of the control qubit in the $|\pm\rangle$-basis. Then we obtain the outcome "$-$" with the probability:

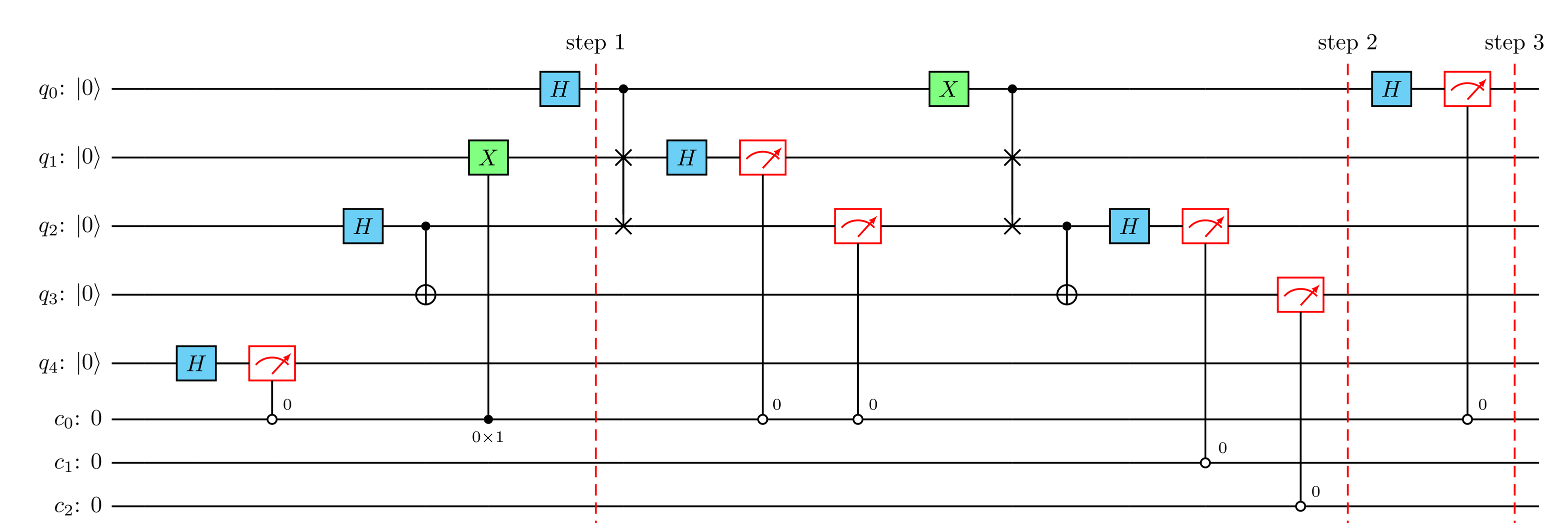
$$p_- = \frac{1}{4d}\sum_{ij}\big\|[P_i,Q_j]\big\|_2^2 = \frac{1}{2}\Big(1 - \frac{1}{d}\sum_{ij}\mathrm{Tr}[P_iQ_jP_iQ_j]\Big),$$

where $\|O\|_2^2 := \mathrm{Tr}[O^\dagger O]$.

We obtain our main result:

$$\mathrm{MED}(A,B) = \sqrt{2p_-}. \qquad (2)$$

## Probabilistic implementation of the quantum SWITCH




**Step 1:** initialization of the registers.

**Step 2:** probabilistic modelling of a CTC.

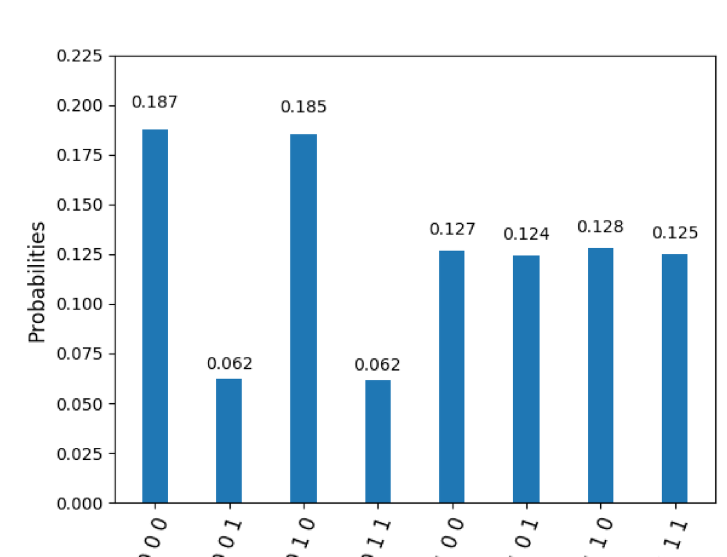
**Step 3:** measurement of the control qubit.

## Performance

The minimum number of repetitions of the experiment needed to estimate MED within an error $\epsilon$ with a probability $1-\delta$ is

$$n(\epsilon,\delta) = -\frac{1}{2\epsilon^2}\log\frac{\delta}{2}, \qquad (3)$$

hence, the sample complexity of the estimation protocol is of $O\left(\frac{1}{\epsilon^2}\log\frac{\delta}{2}\right)$.




**Example:** Consider $A$ and $B$ on MUBs $\{|\pm\rangle\}$ and $\{|0/1\rangle\}$. After running the circuit for 40000 shots, we have obtained

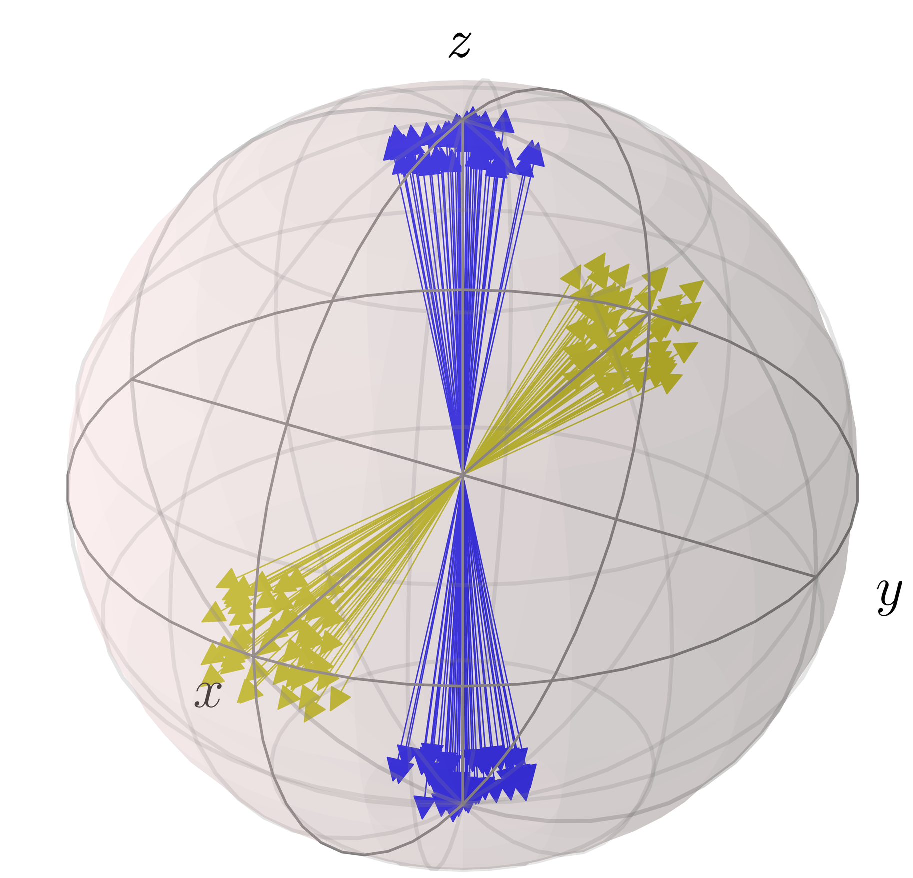
$$\mathrm{MED}(A,B) = \sqrt{\frac{1}{1+\frac{p(000)}{p(001)}}} \approx 0.5,$$

as expected.

## Clustering for projective measurements

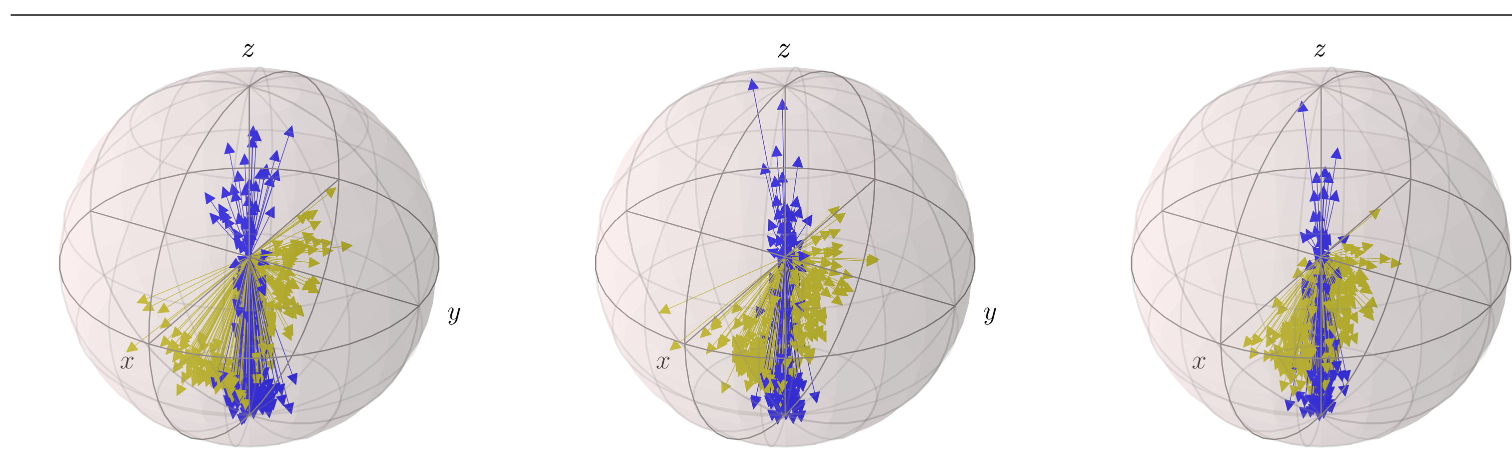
Algorithm:

1) access to $m$ black boxes associated to observables $A_1,\ldots,A_m$,
2) estimation of MED for every pair of observables,
3) clustering of the observables $k$-medoids clustering with $k$-means++ style initial seeding with MED as a distance.




We generate $m = 100$ random qubit observables, of the form $A_l = b_x^{(l)}X + b_y^{(l)}Y + b_z^{(l)}Z$, $l\in\{1,\ldots,m\}$, with $\mathbf{b}_l = (b_x^l, b_y^l, b_z^l) \in^3$ being the Bloch vector of the $l$-th observable. The observables are naturally divided into 2 clusters, as expected.

## Clustering for noisy measurements




Projective measurement $\mathbf{P}^{(l)}$ of each observable is replaced by a non-projective measurement

$$\mathbf{N}^{(l)} = (1-\lambda_l)\,\mathbf{P}^{(l)} + \lambda_l\,\mathbf{T}^{(l)},$$

with $\mathbf{T}^{(l)}$ being a trivial measurement, and random noise probability $\lambda_l = \eta R_l$, where we take noise level $\eta = 0.25,\ 0.5,\ 0.75$.