Compatibility of cyclic causal structures with spacetime in theories with free interventions

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Overview

Causality constitues a pivotal concept in foundational physics. However, there exist different notions:

- **information-theoretic**: representing the information flow between different systems
- **relativistic**: allowing signalling only along non-spacelike curves, i.e. within the light cone of each point

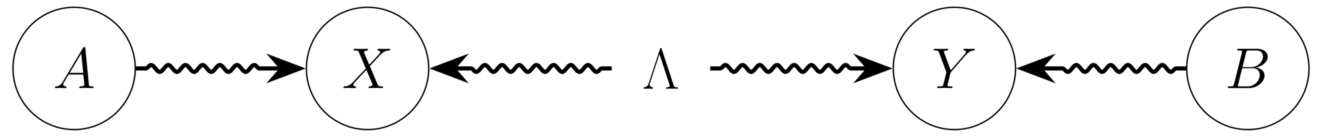
We present a quite generic and theory-independent approach to realize and relate both notions [1][2][3].

Causal Models and Interventions

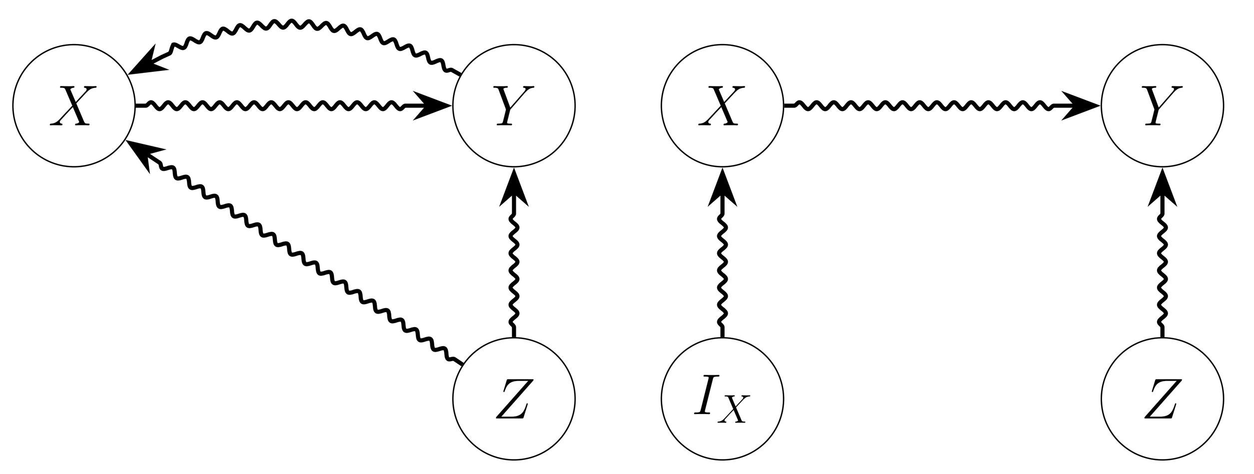
Information-theoretic causal models are given by

- the causal structure, given by a directed graph $\mathcal{G}$, giving dependences (direct causes) and direction of causation
- a probability distribution P_{obs} over random variables (RVs) S, corresponding to the observable nodes

Unobservable nodes Λ can correspond to various objects, e.g. further RVs, quantum states or objects from generalized probabilistic theories (GPTs). Hence, any operational theory can be captured. E.g., consider the standard Bell structure (measurement choices A, B, outcomes X, Y):



Interventions: Enforce a specific value x for (RV associated with) node X by replacing all its parents with additional node I_X. This makes its probability distribution independent from other nodes. Thereby, $\mathcal{G} \mapsto \mathcal{G}_{\text{do}(X)}$. This generalizes to independent interventions for sets of RVs.



Affects Relations

Let $X, Y, Z, W \subset S$, with X, Y non-empty. Then

$$X \models Y \mid \text{do}(Z)$$

if there exist values x of X and z of Z such that

$$P_{\mathcal{G}_{\text{do}(XZ)}}(Y|X = x, Z = z) \neq P_{\mathcal{G}_{\text{do}(Z)}}(Y|Z = z)$$

If $Z \neq \emptyset$, it is a *higher-order (HO) affects relation*. Otherwise, it is a *0th-order affects relation*.

⇒ General model for signalling between RVs!

Properties of Affects Relations

Reducibility: Superfluous nodes in X: $\exists s_X \oplus \tilde{s}_X = X : s_X \not\models Y \mid \text{do}(Z\tilde{s}_X) \implies \tilde{s}_X \models Y \mid \text{do}(Z)$.

Decreasability: Superfluous nodes in Z: $\exists e_Z \in Z : X \models Y \mid \text{do}(Z \setminus e_Z)$.

Causal Inference ($\twoheadrightarrow$ denotes potentially indirect causes):

- $X \models Y \mid \text{do}(Z) \implies X \twoheadrightarrow Y$.
- $X \models Y \mid \text{do}(Z) \wedge X \not\models Y \mid \text{do}(Z \setminus e_Z) \implies e_Z \twoheadrightarrow Y$.
- $X \models Y \mid \text{do}(Z)$ irreducible & indecreasable $\implies e_{XZ} \twoheadrightarrow Y \quad \forall e_{XZ} \in XZ$.

Possible to use *absence* of affects relations for inference!

Affects Causal Loops (ACLs)

Affects Causal Loops are loops in the causal structure that are detectable using the causal inference rules for affects relations. If only a set of present affects relations $\mathscr{A}$ is known, the first rule uncovers all detectable causal relations.

- ACL from a Complete Affects Chain: e.g. $\mathscr{A} = \{AB \models CD, CDE \models F, F \models A\}$.
- ACL from multiple Incomplete Affects Chains: e.g. $\mathscr{A} = \{A \models BC, B \models AC, C \models AB\}$, $\mathscr{A} = \{X \models Y, Y \models AB, A \models X, Z \models AB, B \models Z\}$.

If and only if the graphical representation (shown on the right for the last example) has strongly connected components, it implies the presence of a causal loop.

Spacetime and Embedding

Spacetime is modelled as a partially ordered set (poset) $\mathcal{T}$ of its points, specifying their causal order. Then two distinct points $a, b \in \mathcal{T}$ can be either ordered ($a \prec b$, $a \succ b$) or unordered ($a \not\prec\not\succ b$) with regard to each other.

Each RV X is embedded into $\mathcal{T}$, by specifying

- $O(X) \in \mathcal{T}$, a *location* of the RV in spacetime, without any immediate information-theoretic implications.
- $\mathcal{R}_X \subset \mathcal{T}$, where the RV and the information encoded is operationally *accessible*.

This procedure yields ordered RVs (ORV) $\mathcal{X} := (X, O(X))$, and the *relativistic future* $\bar{\mathcal{F}}(\mathcal{X}) := \{a \in \mathcal{T} | a \succeq O(X)\}$. An embedding is called *degenerate* if any location in $\mathcal{T}$ is shared by multiple RVs: $O(X) = O(Y)$.

Compatibility

Let $\mathcal{S}$ be a set of ORVs from a set of RVs S and a poset $\mathcal{T}$ with an embedding $\mathcal{E}$. Then a set of affects relations $\mathscr{A}$ is *compatible* with $\mathcal{E}$ if the following conditions hold:

1. Let $X, Y, Z \subset S$ disjoint. If $(X \models Y \mid \text{do}(Z)) \in \mathscr{A}$ is irreducible, then $\mathcal{R}_{YZ} = \mathcal{R}_Y \cap \mathcal{R}_Z \subseteq \mathcal{R}_X$.
2. $\mathcal{R}_X = \bar{\mathcal{F}}(\mathcal{X}) \; \forall \mathcal{X} \in \mathcal{S}$ (Broadcasting).

- Stability: Strict subset relation $\mathcal{R}_Y \cap \mathcal{R}_Z \subset \mathcal{R}_X$.

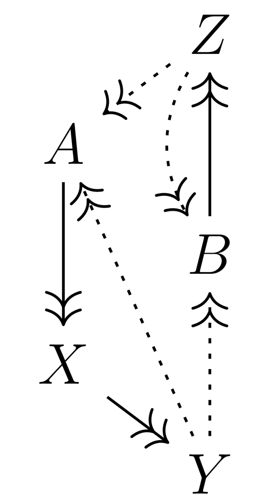
Higher-dim. Minkowski Spacetime

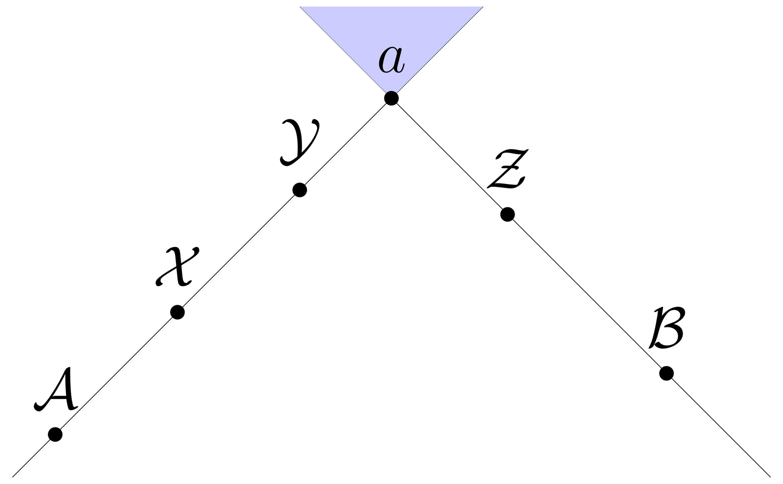
While Minkowski spacetime with 1 spatial dimension forms an *order lattice*, in higher dimensions, other order-theoretic properties are present. In the following, $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subset \mathcal{S}$.

- *Spanning elements*: $\text{span}(\mathcal{X})$ contains all $\mathcal{X}_i \in \mathcal{X}$ which are required to form their joint future $\bar{\mathcal{F}}(\mathcal{X})$.
- *Conicality*: Joint Future of $\mathcal{X}_i$ implies their locations. $\bar{\mathcal{F}}(\mathcal{X}) \mapsto O(\mathcal{X}_i) \; \forall \mathcal{X}_i \in \text{span}(\mathcal{X})$.
- *Location Symmetry*: $\bar{\mathcal{F}}(\mathcal{X}\mathcal{Y}) = \bar{\mathcal{F}}(\mathcal{X}\mathcal{Z}) \implies \bar{\mathcal{F}}(\mathcal{X}) \subseteq \bar{\mathcal{F}}(\mathcal{Y}\mathcal{Z}) \vee \exists s_{\mathcal{Y}}, s_{\mathcal{Z}} \subseteq \mathcal{Y}, \mathcal{Z}\colon \bar{\mathcal{F}}(s_{\mathcal{Y}}) = \bar{\mathcal{F}}(s_{\mathcal{Z}})$.

Compatibility of Affects Causal Loops

Compatible embeddings of causal loops into partially ordered spacetimes $\mathcal{T}$ are possible. While all embeddings of loops from complete affects chains are unstable, the same does not hold for embeddings of incomplete affects chains. There is an example for a stable embedding into 1+1-Minkowski spacetime:





Here still some futures coincide, e.g. $\bar{\mathcal{F}}(\mathcal{Y}\mathcal{Z}) = \bar{\mathcal{F}}(\mathcal{A}\mathcal{B})$, which is in conflict with conicality. Hence, no example of a compatible embedding is known for higher dimensions.

Compatibility and Indecreasability

For irreducible indecreasable affects relations, the higher-order term Z appears on opposite sites for causal inference ($e_{XZ} \twoheadrightarrow Y \; \forall e_{XZ} \in XZ$) and compatibility ($\mathcal{R}_X \subset \mathcal{R}_{YZ}$). If $\mathcal{T}$ shows location symmetry, for any non-degenerate embedding, compatibility implies the condition

- Let $X, Y, Z \subset S$ disjoint. If $(X \models Y \mid \text{do}(Z)) \in \mathscr{A}$ is indecreasable, then $\mathcal{R}_{YX} = \mathcal{R}_Y \cap \mathcal{R}_X \subseteq \mathcal{R}_Z$.

If $\mathcal{T}$ further shows conicality, we get

$$\mathcal{R}_Y \subseteq \mathcal{R}_X \cap \mathcal{R}_Z$$

for any irreducible indecreasable affects relation, restoring the symmetry between causal inference and compatibility.

References

[1] V. Vilasini and R. Colbeck. Physical Review A, 106, p. 032204 (2022).
[2] V. Vilasini and R. Colbeck. Physical Review Letters, 129, p. 110401 (2022).
[3] M. Grothus. Masters thesis. ETH Zurich, Sept. 2022.